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NOETHER'S PROBLEM FOR ABELIAN EXTENSIONS OF CYCLIC p-GROUPS II

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Abstract. Let K be a field and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. Noether's problem then asks whether $K(G)$ is rational (i.e., purely transcendental) over K. In this article we prove three results that give a positive answer to Noether's problem (under suitable conditions) for a number of p-groups G that are abelian extensions of cyclic p-groups. In particular, our results hold for all basic types of groups with three generators having this property.

1. Introduction

Let K be any field. A field extension L of K is called rational over K (or K-rational, for short) if $L \simeq K(x_1, \dots, x_n)$ over K for some integer n, with $x_1, \dots, x_n$ algebraically independent over K. Now let G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. *Noether's problem* then asks whether $K(G)$ is rational over K. This is related to the inverse Galois problem, to the existence of generic G-Galois extensions over K, and to the existence of versal G-torsors over K-rational field extensions [Sw, Sa1, GMS, 33.1, p.86]. Noether's problem for abelian groups was studied extensively by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. The reader is referred to Swan's paper for a survey of this problem [Sw]. Fischer's Theorem is a starting point of investigating Noether's problem for finite abelian groups in general.

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Theorem 1.1. (Fischer [Sw, Theorem 6.1]) *Let G be a finite abelian group of exponent e. Assume that* (i) *either char $K = 0$ or char $K > 0$ with char $K \nmid e$, and* (ii) *K contains a primitive e-th root of unity. Then $K(G)$ is rational over K.*

The next stage is the investigation of Noether's problem for finite meta-abelian groups, and in particular metacyclic p-groups. Recall that any metacyclic p-group G is generated by two elements σ and τ with relations $\sigma^{p^a} = 1, \tau^{p^b} = \sigma^{p^c}$ and $\tau^{-1}\sigma\tau = \sigma^{\varepsilon+\delta p^r}$ where $\varepsilon = 1$ if p is odd, $\varepsilon = \pm 1$ if $p = 2$, $\delta = 0, 1$ and $a, b, c, r \geq 0$ are subject to some restrictions. For the the description of these restrictions see e.g. [Ka1, p. 564]. The following Theorem of Kang generalizes Fischer's Theorem for the metacyclic p-groups.

Theorem 1.2. (Kang[Ka1, Theorem 1.5]) *Let G be a metacyclic p-group with exponent p^e, and let K be any field such that* (i) *char $K = p$, or* (ii) *char $K \neq p$ and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

Other results of Noether's problem for p-groups the reader can find in [CK, HuK, Ka2]. In a recent paper [Mi] we proved two results for p-groups that are abelian extensions of cyclic p-groups.

Theorem 1.3. (Michailov[Mi, Theorem 1.8]) *Let G be a group of order p^n for $n \geq 2$ with an abelian subgroup H of order p^{n-1}, and let G be of exponent p^e. Choose any $\alpha \in G$ such that α generates G/H, i.e., $\alpha \notin H, \alpha^p \in H$. Denote $H(p) = \{h \in H : h^p = 1, h \notin H^p\}$, and assume that $[H(p), \alpha] \subset H(p)$. Denote by $G_{(i)} = [G, G_{(i-1)}]$ the lower central series for $i \geq 1$ and $G_{(0)} = G$. Let the p-th lower central subgroup $G_{(p)}$ be trivial. Assume that* (i) *char $K = p > 0$, or* (ii) *char $K \neq p$ and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

Theorem 1.4. (Michailov[Mi, Theorem 1.9]) *Let G be a group of order p^n for $n \leq 6$ with an abelian normal subgroup H, such that G/H is cyclic. Let G be of exponent p^e. Assume that* (i) *char $K = p > 0$, or* (ii) *char $K \neq p$ and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

The purpose of this paper is to prove a generalization of the latter two Theorems for several series of p-groups that are abelian extensions of cyclic p-groups. However,

we should not "over-generalize" Theorem 1.4 to the case of any meta-abelian group because of the following Theorem of Saltman.

Theorem 1.5. (Saltman [Sa2]) *For any prime number p and for any field K with char $K \neq p$ (in particular, K may be an algebraically closed field), there is a meta-abelian p-group G of order p^9 such that $K(G)$ is not rational over K.*

Let p be an odd prime, let C_{p^a} be a cyclic group of order p^a generated by the element α, and let H be an abelian group generated by two elements β and γ having orders p^b and p^c, respectively. Assume that G is any group extension of C_{p^a} by H, i.e., G is in the middle of the exact sequence $1 \to H \to G \to C_{p^a} \to 1$. Our first goal is to investigate Noether's problem for any group extension G such that $\langle \beta \rangle$ is normal in G. Then the subgroup generated by β and α is metacyclic, so $[\beta, \alpha] = \beta^{\varepsilon_1 p^r}$ for some $r \geq 1, \varepsilon_1 \in \{0, 1\}$. The quotient group $G/\langle \beta \rangle$ is also metacyclic, so $[\gamma, \alpha] = \beta^x \gamma^{\varepsilon_2 p^s}$ for some $\varepsilon_2 \in \{0, 1\}, x \in \mathbb{Z}$ and $s \geq 1$.

The first main result of this paper is the following Theorem.

Theorem 1.6. *Let p be an odd prime, and let G be a p-group generated by three elements α, β and γ such that $\alpha^{p^a} \in \langle \beta, \gamma \rangle, \beta^{p^b} = \gamma^{p^c} = 1, [\beta, \gamma] = 1, [\beta, \alpha] = \beta^{\varepsilon_1 p^r}, [\gamma, \alpha] = \beta^x \gamma^{\varepsilon_2 p^s}$ for some $a, b, c \in \mathbb{N}, \varepsilon_1, \varepsilon_2 \in \{0, 1\}, r \geq 1, s \geq 1, x \in \mathbb{Z}$. If $\varepsilon_1 = \varepsilon_2 = 1$ assume additionally that $\gcd(x, p) = 1$ and $r + s \geq \max\{b, c\}$. Denote by p^e the exponent of G. Assume that* (i) *char $K = p > 0$, or* (ii) *char $K \neq p$ and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

The key idea to prove Theorem 1.6 is to find a faithful G-subspace W of the regular representation space $\bigoplus_{g \in G} K \cdot x(g)$ and to show that W^G is rational over K. The subspace W is obtained as an induced representation from H. By applying various linearizing techniques we then reduce the rationality problem to another rationality problem which is related to some properly constructed metacyclic p-group.

We will also investigate groups with relations $[\beta, \alpha] = \gamma^{p^s}$ and $[\gamma, \alpha] = \beta^{p^{b-a}} \gamma^x$ where $b - a + s \geq \max\{b, c\}$. Then $\beta^{p^{b-a}}$ is central and the subgroup $\langle \gamma, \alpha \rangle \leq G/\langle \beta^{p^{b-a}} \rangle$ is metacyclic. Therefore, $x = 0$ or $x = p^t$ for some $t \geq 1$. The second main result of this paper is the following Theorem.

Theorem 1.7. *Let p be an odd prime, and let G be a p-group generated by three elements α, β and γ such that $\alpha^{p^a} \in \langle \beta, \gamma \rangle, \beta^{p^b} = \gamma^{p^c} = 1, [\beta, \gamma] = 1, [\beta, \alpha] = \gamma^{p^s}, [\gamma, \alpha] = \beta^{p^r}\gamma^{\varepsilon p^t}$ for some $a, b, c \in \mathbb{N}, \varepsilon \in \{0, 1\}, r = b - a \geq 1, s \geq 1, t \geq 1$. If $\varepsilon = 0$ assume additionally that $r + s \geq \max\{b, c\}$. If $\varepsilon = 1$ assume additionally that either $s \leq t$ or $c \leq \min\{2t, a + t\}$. Denote by p^e the exponent of G. Assume that* (i) *char $K = p > 0$, or* (ii) *char $K \neq p$ and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

Our main theorems give us a clue to find an answer to Noether's problem for groups G with more than three generators.

Corollary 1.8. *Let p be an odd prime, and let G be a p-group generated by the elements $\alpha_1, \ldots, \alpha_s$ (for $s \geq 2$) and α such that $\alpha^{p^a} \in \langle \alpha_1, \ldots, \alpha_s \rangle, \alpha_i^{p^{a_i}} = 1, [\alpha_i, \alpha_j] = 1, [\alpha_i, \alpha] = \alpha_{i+1}^{p^{r_{i+1}}}, [\alpha_s, \alpha] = \alpha_1^{\varepsilon_1 r_1}\alpha_s^{\varepsilon_2 r_{s+1}}$, where $a_i - r_i = a, 2 \leq i \leq s, r_i + r_j \geq \max\{a_1, \ldots, a_s\}, 1 \leq i < j \leq s + 1, \varepsilon_1, \varepsilon_2 \in \{0, 1\}$. If $\varepsilon_2 = 1$ assume additionally that $r_{s+1} \geq r_s$. Denote by p^e the exponent of G. Assume that* (i) *char $K = p > 0$, or* (ii) *char $K \neq p$ and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

Throughout this article we are trying to keep the inequalities related to the generators as weak as possible, since this would give us a broader range of groups having a positive answer to Noether's problem. However, in order to simplify the notations in the proof, when dealing with groups with more than three generators as in Corollary 1.8, we demand also the inequality $r_1 + r_{s+1} \geq \max\{a_1, \ldots, a_s\}$.

We organize this paper as follows. We recall some preliminaries in Section 2 and also give the outline of a generalization of [Ka1, Theorem 4.1]. These results play an important role in the proof of Theorem 1.6, which is given in Section 3. The proofs of Theorem 1.7 and Corollary 1.8 are given respectively in Sections 4 and 5.

2. Preliminaries

We list several results which will be used in the sequel.

Theorem 2.1. ([HK, Theorem 1]) *Let G be a finite group acting on $L(x_1, \ldots, x_m)$, the rational function field of m variables over a field L such that*

- **(i):** *for any $\sigma \in G, \sigma(L) \subset L$;*

(ii): *the restriction of the action of G to L is faithful;*

(iii): *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in \mathrm{GL}_m(L)$ and $B(\sigma)$ is $m \times 1$ matrix over L. Then there exist $z_1, \ldots, z_m \in L(x_1, \ldots, x_m)$ so that $L(x_1, \ldots, x_m)^G = L^G(z_1, \ldots, z_m)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \le i \le m$.

Theorem 2.2. ([AHK, Theorem 3.1]) *Let G be a finite group acting on $L(x)$, the rational function field of one variable over a field L. Assume that, for any $\sigma \in G, \sigma(L) \subset L$ and $\sigma(x) = a_\sigma x + b_\sigma$ for any $a_\sigma, b_\sigma \in L$ with $a_\sigma \neq 0$. Then $L(x)^G = L^G(z)$ for some $z \in L[x]$.*

Theorem 2.3. ([CK, Theorem 1.7]) *If $\operatorname{char} K = p > 0$ and $\widetilde{G}$ is a finite p-group, then $K(G)$ is rational over K.*

The following Lemma can be extracted from some proofs in [Ka2, HuK].

Lemma 2.4. *Let $\langle\tau\rangle$ be a cyclic group of order $n > 1$, acting on $K(v_1, \ldots, v_{n-1})$, the rational function field of $n-1$ variables over a field K such that*

$$\tau \ : \ v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{n-1} \mapsto (v_1 \cdots v_{n-1})^{-1} \mapsto v_1.$$

If K contains a primitive n-th root of unity ξ, then $K(v_1, \ldots, v_{n-1}) = K(s_1, \ldots, s_{n-1})$ where $\tau : s_i \mapsto \xi^i s_i$ for $1 \le i \le n-1$.

Proof. Define $w_0 = 1 + v_1 + v_1 v_2 + \cdots + v_1 v_2 \cdots v_{n-1}, w_1 = (1/w_0) - 1/n, w_{i+1} = (v_1 v_2 \cdots v_i / w_0) - 1/n$ for $1 \le i \le n-1$. Thus $K(v_1, \ldots, v_{n-1}) = K(w_1, \ldots, w_n)$ with $w_1 + w_2 + \cdots + w_n = 0$ and

$$\tau \ : \ w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{n-1} \mapsto w_n \mapsto w_1.$$

Define $s_i = \sum_{1 \le j \le n} \xi^{-ij} w_j$ for $1 \le i \le n-1$. Then $K(w_1, \ldots, w_n) = K(s_1, \ldots, s_{n-1})$ and $\tau : s_i \mapsto \xi^i s_i$ for $1 \le i \le n-1$. $\square$

Lemma 2.5. (Kang [Ka1, Lemma 2.4]) *Let p be a prime number, n, r, i be positive integers with $p \le i \le p^n$. Write $p^l \le i < p^{l+1}$ for some positive integer l. In case $p = 2$ assume furthermore that $l \ge 2$. Then $\binom{p^n}{i} p^{ir}$ is divisible by p^{n+r+1}.*

We can now generalize [Ka1, Lemma 2.5].

Lemma 2.6. *Let p be a prime number, let n, r and t be positive integers, and let $a = 1 + b \cdot p^t$, where $b \in \{-1, 0, 1\}$.*

(i): *If $(p, r) \neq (2, 1)$, then $(1 + a \cdot p^r)^{p^n} = 1 + a_1 \cdot p^{n+r}$ for some integer a_1 with $p \nmid a_1$.*

(ii): *If $r \geq 2$, then $(-1 + a \cdot 2^r)^{2^n} = 1 + a_1 \cdot 2^{n+r}$ and $(1 + a \cdot 2)^{2^n} = 1 + b_1 \cdot 2^{n+2}$ where a_1 and b_1 are odd integers.*

Proof. Apply Lemma 2.5 for (i) and $(-1 + a \cdot 2^r)^{2^n}$. Since $(1 + a \cdot 2)^{2^n} = (-1 + 2^2(1 + b \cdot 2^{t-1}))^{2^n}$, we may apply the result $(-1 + a \cdot 2^r)^{2^n}$ with $r \geq 2$. □

The proof of the following key result now can be obtained by repeating literally the proof of [Ka1, Theorem 4.1] (including an analog of [Ka1, Lemma 4.2], for which we can apply Lemma 2.6).

Theorem 2.7. *Let G be a metacyclic p-group of exponent p^e, generated by σ and τ such that $\sigma^{p^m} = \tau^{p^n} = 1, \tau^{-1}\sigma\tau = \sigma^s$ for $s = \varepsilon + ap^r$, where $a = 1 + b \cdot p^t$ for $t \in \mathbb{N}$ and $b \in \{-1, 0, 1\}$. Let K be a field, containing a primitive p^e-th root of unity, and let ζ be a primitive p^m-th root of unity. Then $K(u_0, u_1, \ldots, u_{p^n-1})^G$ is rational over K, where G acts on $u_0, \ldots, u_{p^n-1}$ by*

$$\sigma \ : \ u_i \mapsto \zeta^{s^i} u_i,$$

$$\tau \ : \ u_0 \mapsto u_1 \mapsto \cdots \mapsto u_{p^n-1} \mapsto u_0.$$

Notice that there indeed exist metacyclic p-groups with the relations given in the latter Theorem. These relations do not contradict with the standard relations described in Section 1 (which are derived from the classification Theorem for the metacyclic p-groups). Although in the proof of Cases I-III in Section 3 we will need only [Ka1, Theorem 4.1], in the proof of Case IV we will need the more general Theorem 2.7.

3. Proof of Theorem 1.6

Let V be a K-vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in G} K \cdot x(g)$ where G acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in G$. Thus $K(V)^G = K(x(g) : g \in G)^G = K(G)$. The key idea is to find a faithful G-subspace W of V^* and to show

that W^G is rational over K. The subspace W is obtained as an induced representation from H.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_{i=0}^{p^b-1} x(\beta^i),\ \ X_2 = \sum_{i=0}^{p^c-1} x(\gamma^i).$$

Note that $\beta \cdot X_1 = X_1$ and $\gamma \cdot X_2 = X_2$.

Let $\zeta_{p^b} \in K$ be a primitive p^b-th root of unity, and let $\zeta_{p^c} \in K$ be a primitive p^c-th root of unity. Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^c-1} \zeta_{p^c}^{-1}\gamma^i \cdot X_1,\ \ Y_2 = \sum_{i=0}^{p^b-1} \zeta_{p^b}^{-1}\beta^i \cdot X_2.$$

It follows that

$$\begin{aligned} \beta\ &: Y_1 \mapsto Y_1,\ \ Y_2 \mapsto \zeta_{p^b} Y_2, \\ \gamma\ &: Y_1 \mapsto \zeta_{p^c} Y_1,\ \ Y_2 \mapsto Y_2. \end{aligned}$$

Thus $K\cdot Y_1 + K\cdot Y_2$ is a representation space of the subgroup H. The induced subspase W depends on the relations in G. It is well known that the case $\alpha^{p^a} = \beta^f\gamma^h$ can easily be reduced to the case $\alpha^{p^a} = 1$ (see e.g. [Mi, Proof of Theorem 1.8, Step 2]). We have four types of presentations for the group G, so we will consider separately these four cases.

Case I. $[\beta, \alpha] = 1$ and $[\gamma, \alpha] = \beta^x$ for some $x \in \mathbb{Z}$.

Define $x_i = \alpha^i \cdot Y_1, y_i = \alpha^i \cdot Y_2$ for $0 \le i \le p^a - 1$. Note that from the relation $[\gamma, \alpha] = \beta^x$ it follows that $\gamma\alpha^i = \alpha^i\gamma\beta^{ix}$. We have now

$$\begin{aligned} \beta\ &: x_i \mapsto x_i,\ \ y_i \mapsto \zeta_{p^b} y_i, \\ \gamma\ &: x_i \mapsto \zeta_{p^c} x_i,\ \ y_i \mapsto \zeta_{p^b}^{ix} y_i, \\ \alpha\ &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^a-1} \mapsto x_0, \\ &\quad\ y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^a-1} \mapsto y_0. \end{aligned}$$

for $0 \le i \le p^a - 1$.

For $1 \le i \le p^a - 1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \le i \le p^a - 1) = K(x_0, y_0, u_i, v_i : 1 \le i \le p^a - 1)$ and for every $g \in G$

$$g \cdot x_0 \in K(u_i, v_i : 1 \le i \le p^a - 1) \cdot x_0,\ \ g \cdot y_0 \in K(u_i, v_i : 1 \le i \le p^a - 1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \le i \le p^a - 1)$ is invariant by the action of G. Thus $K(x_i, y_i : 0 \le i \le p^a - 1)^G = K(u_i, v_i : 1 \le i \le p^a - 1)^G(u, v)$ for some u, v such that $\alpha(v) = \beta(v) = \gamma(v) = v$ and $\alpha(u) = \beta(u) = \gamma(u) = u$. We have now

$$\begin{aligned}
&\beta \ : u_i \mapsto u_i,\ v_i \mapsto v_i,\\
&\gamma \ : u_i \mapsto u_i,\ v_i \mapsto \zeta_{p^b}^x v_i, \qquad\qquad (3.1)\\
&\alpha \ : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^a-1} \mapsto (u_1 u_2 \cdots u_{p^a-1})^{-1},\\
&\qquad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^a-1} \mapsto (v_1 v_2 \cdots v_{p^a-1})^{-1},
\end{aligned}$$

for $1 \le i \le p^a - 1$. From Theorem 2.2 it follows that if $K(u_i, v_i : 1 \le i \le p^a - 1)^G$ is rational over K, so is $K(x_i, y_i : 0 \le i \le p^a - 1)^G$ over K.

Now, consider the metacyclic p-group $\widetilde{G} = \langle \sigma, \tau : \sigma^{p^{2b}} = \tau^{p^a} = 1, \tau^{-1}\sigma\tau = \sigma^k, k = 1 + p^b \rangle$.

Define $X = \sum_{0 \le j \le p^{2b}-1} \zeta_{p^{2b}}^{-j} x(\sigma^j), V_i = \tau^i X$ for $0 \le i \le p^a - 1$. It follows that

$$\begin{aligned}
\sigma \ &: \ V_i \mapsto \zeta_{p^{2b}}^{k^i} V_i,\\
\tau \ &: \ V_0 \mapsto V_1 \mapsto \cdots \mapsto V_{p^a-1} \mapsto V_0.
\end{aligned}$$

Note that $K(V_0, V_1, \dots, V_{p^a-1})^{\widetilde{G}}$ is rational by Theorem 2.7.

Define $U_i = V_i / V_{i-1}$ for $1 \le i \le p^a - 1$. Then $K(V_0, V_1, \dots, V_{p^a-1})^{\widetilde{G}} = K(U_1, U_2, \dots, U_{p^a-1})^{\widetilde{G}}(U)$ where

$$\begin{aligned}
\sigma \ &: \ U_i \mapsto \zeta_{p^{2b}}^{k^i - k^{i-1}} U_i,\ U \mapsto U\\
\tau \ &: \ U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^a-1} \mapsto (U_1 U_2 \cdots U_{p^a-1})^{-1},\ U \mapsto U.
\end{aligned}$$

Notice that $\zeta_{p^{2b}}^{k^i - k^{i-1}} = \zeta_{p^{2b}}^{(1+p^b)^{i-1} p^b} = \zeta_{p^b}^{(1+p^b)^{i-1}} = \zeta_{p^b}$. Compare Formula (3.1) (i.e., the actions of γ and α on $K(v_i : 1 \le i \le p^a - 1)$) with the actions of $\widetilde{G}$ on $K(U_i : 1 \le i \le p^a - 1)$. They are almost the same. (In fact, the value of x is of no importance, so we we can assume that $x = 1$.) Hence, according to Theorem 2.7, we get that $K(v_1, \dots, v_{p^a-1})^G(v) \cong K(U_1, \dots, U_{p^a-1})^{\widetilde{G}}(U) = K(V_0, V_1, \dots, V_{p^a-1})^{\widetilde{G}}$ is rational over K. Since by Lemma 2.4 we can linearize the action of α on $K(u_i : 1 \le i \le p^a - 1)$, we obtain finally that $K(u_i, v_i : 1 \le i \le p^a - 1)^{\langle \gamma, \alpha \rangle}$ is rational over K.

Case II. $[\beta, \alpha] = 1$ and $[\gamma, \alpha] = \beta^x \gamma^{p^s}$ for some $x \in \mathbb{Z}, s \in \mathbb{N}$.

Define $x_i = \alpha^i \cdot Y_1, y_i = \alpha^i \cdot Y_2$ for $0 \le i \le p^a - 1$. Calculations show that $\gamma\alpha^i = \alpha^i \gamma^{k(i)} \beta^{xl(i)}$ for $k(i) = 1 + \binom{i}{1} p^s + \binom{i}{2} p^{2s} + \cdots + \binom{i}{i} p^{is} = k^i$, where $k = 1 + p^s$, and

$l(i) = \binom{i}{1} + \binom{i}{2}p^s + \cdots + \binom{i}{i}p^{(i-1)s}$. We have now

$$\begin{aligned}
\beta\ &: x_i \mapsto x_i,\ y_i \mapsto \zeta_{p^b} y_i,\\
\gamma\ &: x_i \mapsto \zeta_{p^c}^{k^i} x_i,\ y_i \mapsto \zeta_{p^b}^{xl(i)} y_i,\\
\alpha\ &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^a-1} \mapsto x_0,\\
&\quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^a-1} \mapsto y_0.
\end{aligned}$$

for $0 \le i \le p^a - 1$.

For $1 \le i \le p^a - 1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \le i \le p^a - 1) = K(x_0, y_0, u_i, v_i : 1 \le i \le p^a - 1)$ and for every $g \in G$

$$g \cdot x_0 \in K(u_i, v_i : 1 \le i \le p^a - 1) \cdot x_0,\ g \cdot y_0 \in K(u_i, v_i : 1 \le i \le p^a - 1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \le i \le p^a - 1)$ is invariant by the action of G. Thus $K(x_i, y_i : 0 \le i \le p^a - 1)^G = K(u_i, v_i : 1 \le i \le p^a - 1)^G(u, v)$ for some u, v such that $\alpha(v) = \beta(v) = \gamma(v) = v$ and $\alpha(u) = \beta(u) = \gamma(u) = u$. Notice that $l(i) - l(i-1) = k^{i-1}$. We have now

$$\begin{aligned}
\beta\ &: u_i \mapsto u_i,\ v_i \mapsto v_i,\\
\gamma\ &: u_i \mapsto \zeta_{p^c}^{p^s k^{i-1}} u_i,\ v_i \mapsto \zeta_{p^b}^{xk^{i-1}} v_i,\\
\alpha\ &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^a-1} \mapsto (u_1 u_2 \cdots u_{p^a-1})^{-1},\\
&\quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^a-1} \mapsto (v_1 v_2 \cdots v_{p^a-1})^{-1},
\end{aligned} \tag{3.2}$$

for $1 \le i \le p^a - 1$. From Theorem 2.2 it follows that if $K(u_i, v_i : 1 \le i \le p^a - 1)^G$ is rational over K, so is $K(x_i, y_i : 0 \le i \le p^a - 1)^G$ over K.

We can always write x in the form $x = yp^t$ for $y \in \mathbb{Z} : \gcd(y, p) = 1$ and $t \ge 0$. Assume that $b - t \le c - s$. (The other case $b - t > c - s$ is identical, so we leave it to the reader.) For $1 \le i \le p^a - 1$, define $w_i = v_i / u_i^{yp^{c-s-b+t}}$. Then $K(u_i, v_i : 1 \le i \le p^a - 1) = K(u_i, w_i : 1 \le i \le p^a - 1)$ and $\gamma(w_i) = w_i$ for $1 \le i \le p^a - 1$.

Now, consider the metacyclic p-group $\widetilde{G} = \langle \sigma, \tau : \sigma^{p^c} = \tau^{p^a} = 1, \tau^{-1}\sigma\tau = \sigma^k, k = 1 + p^s \rangle$.

Define $X = \sum_{0 \le j \le p^c - 1} \zeta_{p^c}^{-j} x(\sigma^j), V_i = \tau^i X$ for $0 \le i \le p^a - 1$. It follows that

$$\begin{aligned}
\sigma\ &:\ V_i \mapsto \zeta_{p^c}^{k^i} V_i,\\
\tau\ &:\ V_0 \mapsto V_1 \mapsto \cdots \mapsto V_{p^a-1} \mapsto V_0.
\end{aligned}$$

Note that $K(V_0, V_1, \ldots, V_{p^a-1})^{\widetilde{G}}$ is rational by Theorem 2.7.

Define $U_i = V_i/V_{i-1}$ for $1 \le i \le p^a - 1$. Then $K(V_0, V_1, \ldots, V_{p^a-1})^{\widetilde{G}} = K(U_1, U_2, \ldots, U_{p^a-1})^{\widetilde{G}}(U)$ where

$$\begin{aligned} \sigma &: U_i \mapsto \zeta_{p^c}^{k^i - k^{i-1}} U_i,\ U \mapsto U \\ \tau &: U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^a-1} \mapsto (U_1 U_2 \cdots U_{p^a-1})^{-1},\ U \mapsto U. \end{aligned}$$

Notice that $\zeta_{p^c}^{k^i-k^{i-1}} = \zeta_{p^c}^{p^s k^{i-1}}$. Compare Formula (3.2) (i.e., the actions of γ and α on $K(u_i : 1 \le i \le p^a - 1)$) with the actions of $\widetilde{G}$ on $K(U_i : 1 \le i \le p^a - 1)$. They are the same. Hence, according to Theorem 2.7, we get that $K(u_1, \ldots, u_{p^a-1})^G(u) \cong K(U_1, \ldots, U_{p^a-1})^{\widetilde{G}}(U) = K(V_0, V_1, \ldots, V_{p^a-1})^{\widetilde{G}}$ is rational over K. Since by Lemma 2.4 we can linearize the action of α on $K(w_i : 1 \le i \le p^a - 1)$, we obtain finally that $K(u_i, w_i : 1 \le i \le p^a - 1)^{\langle \gamma, \alpha \rangle}$ is rational over K.

Case III. $[\beta, \alpha] = \beta^{p^r}$ and $[\gamma, \alpha] = \beta^x$ for some $x \in \mathbb{Z}, r \in \mathbb{N}$.

Define $x_i = \alpha^i \cdot Y_1, y_i = \alpha^i \cdot Y_2$ for $0 \le i \le p^a - 1$. Calculations show that $\gamma\alpha^i = \alpha^i \gamma \beta^{xl(i)}$ for $l(i) = \binom{i}{1} + \binom{i}{2} p^r + \cdots + \binom{i}{i} p^{(i-1)r}$, and $\beta\alpha^i = \alpha^i \beta^{k(i)}$ for $k(i) = 1 + \binom{i}{1} p^r + \binom{i}{2} p^{2r} + \cdots + \binom{i}{i} p^{ir} = k^i$, where $k = 1 + p^r$. We have now

$$\begin{aligned} \beta &: x_i \mapsto x_i,\ y_i \mapsto \zeta_{p^b}^{k^i} y_i, \\ \gamma &: x_i \mapsto \zeta_{p^c} x_i,\ y_i \mapsto \zeta_{p^b}^{xl(i)} y_i, \\ \alpha &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^a-1} \mapsto x_0, \\ & \quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^a-1} \mapsto y_0. \end{aligned}$$

for $0 \le i \le p^a - 1$.

For $1 \le i \le p^a - 1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \le i \le p^a - 1) = K(x_0, y_0, u_i, v_i : 1 \le i \le p^a - 1)$ and for every $g \in G$

$$g \cdot x_0 \in K(u_i, v_i : 1 \le i \le p^a - 1) \cdot x_0,\ g \cdot y_0 \in K(u_i, v_i : 1 \le i \le p^a - 1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \le i \le p^a - 1)$ is invariant by the action of G. Thus $K(x_i, y_i : 0 \le i \le p^a - 1)^G = K(u_i, v_i : 1 \le i \le p^a - 1)^G(u, v)$ for some u, v such that $\alpha(v) = \beta(v) = \gamma(v) = v$ and $\alpha(u) = \beta(u) = \gamma(u) = u$. Notice that $l(i) - l(i-1) = k^{i-1}$. We have now

$$\begin{aligned} \beta &: u_i \mapsto u_i,\ v_i \mapsto \zeta_{p^b}^{p^r k^{i-1}} v_i, \\ \gamma &: u_i \mapsto u_i,\ v_i \mapsto \zeta_{p^b}^{x k^{i-1}} v_i, \end{aligned}$$

$$\alpha\ : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^a-1} \mapsto (u_1u_2\cdots u_{p^a-1})^{-1},$$
$$v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^a-1} \mapsto (v_1v_2\cdots v_{p^a-1})^{-1},$$

for $1 \le i \le p^a-1$. From Theorem 2.2 it follows that if $K(u_i, v_i : 1 \le i \le p^a-1)^G$ is rational over K, so is $K(x_i, y_i : 0 \le i \le p^a-1)^G$ over K.

We can always write x in the form $x = yp^t$ for $y \in \mathbb{Z} : \gcd(y,p) = 1$ and $t \ge 0$. Assume that $r \le t$. (The other case $r > t$ is identical, so we leave it to the reader.) Since γ acts in the same way as $\beta^{yp^{t-r}}$ on $K(u_i, v_i : 1 \le i \le p^a-1)$, we find that $K(u_i, v_i : 1 \le i \le p^a-1)^G = K(u_i, v_i : 1 \le i \le p^2-1)^{\langle\beta,\alpha\rangle}$. The proof henceforth is the same as Case II.

Case IV. $[\beta,\alpha] = \beta^{p^r}$ and $[\gamma,\alpha] = \beta^x\gamma^{p^s}$ for some $x \in \mathbb{Z}, \gcd(x,p) = 1, r, s \in \mathbb{N}$, $r+s \ge \max\{b,c\}$.

Define $x_i = \alpha^i \cdot Y_1, y_i = \alpha^i \cdot Y_2$ for $0 \le i \le p^a-1$. Calculations show that $\gamma\alpha^i = \alpha^i\gamma^{k(i)}\beta^{xl(i)}$ for $k(i) = k^i$, $k = 1+p^s$, $l(i) = \binom{i}{1} + \binom{i}{2}(p^r+p^s) + \cdots + \binom{i}{i}(p^{(i-1)r}+p^{(i-1)s})$, and $\beta\alpha^i = \alpha^i\beta^{m(i)}$ for $m(i) = m^i$, where $m = 1+p^r$. We have now

$$\beta\ : x_i \mapsto x_i,\ y_i \mapsto \zeta_{p^b}^{m^i} y_i,$$
$$\gamma\ : x_i \mapsto \zeta_{p^c}^{k^i} x_i,\ y_i \mapsto \zeta_{p^b}^{xl(i)} y_i,$$
$$\alpha\ : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^a-1} \mapsto x_0,$$
$$y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^a-1} \mapsto y_0.$$

for $0 \le i \le p^a-1$.

For $1 \le i \le p^a-1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \le i \le p^a-1) = K(x_0, y_0, u_i, v_i : 1 \le i \le p^a-1)$ and for every $g \in G$

$$g \cdot x_0 \in K(u_i, v_i : 1 \le i \le p^a-1) \cdot x_0,\ g \cdot y_0 \in K(u_i, v_i : 1 \le i \le p^a-1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \le i \le p^a-1)$ is invariant by the action of G. Thus $K(x_i, y_i : 0 \le i \le p^a-1)^G = K(u_i, v_i : 1 \le i \le p^a-1)^G(u,v)$ for some u, v such that $\alpha(v) = \beta(v) = \gamma(v) = v$ and $\alpha(u) = \beta(u) = \gamma(u) = u$. Notice that $l(i) - l(i-1) \equiv (1+p^r+p^s)^{i-1} \pmod{p^b}$. Put $l = 1+p^r+p^s$. We have now

$$\beta\ : u_i \mapsto u_i,\ v_i \mapsto \zeta_{p^b}^{p^r m^{i-1}} v_i,$$
$$\gamma\ : u_i \mapsto \zeta_{p^c}^{p^s k^{i-1}} u_i,\ v_i \mapsto \zeta_{p^b}^{xl^{i-1}} v_i,$$
$$\alpha\ : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^a-1} \mapsto (u_1u_2\cdots u_{p^a-1})^{-1},$$

$$v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^a-1} \mapsto (v_1 v_2 \cdots v_{p^a-1})^{-1},$$

for $1 \le i \le p^a - 1$. From Theorem 2.2 it follows that if $K(u_i, v_i : 1 \le i \le p^a - 1)^G$ is rational over K, so is $K(x_i, y_i : 0 \le i \le p^a - 1)^G$ over K.

Since $r + s \ge c$ and $l^{i-1}p^r = (m + p^s)^{i-1}p^r \equiv m^{i-1}p^r \pmod{p^b}$, we get that β acts in the same way as γ^{yp^r} on $K(u_i, v_i : 1 \le i \le p^a - 1)$ for some $y \in \mathbb{Z} : yx \equiv 1 \pmod{p^b}$. Hence $K(u_i, v_i : 1 \le i \le p^a - 1)^G = K(u_i, v_i : 1 \le i \le p^2 - 1)^{\langle\gamma,\alpha\rangle}$.

Subcase IV.1 Assume that $c - s \le b$. We have that $l^{i-1}p^s = (k + p^r)^{i-1}p^s \equiv k^{i-1}p^s \pmod{p^b}$. For $1 \le i \le p^a - 1$, define $w_i = u_i/v_i^{yp^{s+b-c}}$. Then $K(u_i, v_i : 1 \le i \le p^a - 1) = K(w_i, v_i : 1 \le i \le p^a - 1)$ and $\gamma(w_i) = w_i$ for $1 \le i \le p^a - 1$.

Sub-subcase IV.1.1 Let $s \ge r$. Then $l = 1 + u \cdot p^r$, where $u = 1 + p^{s-r}$. Consider the metacyclic p-group $\widetilde{G} = \langle \sigma, \tau : \sigma^{p^{b+r}} = \tau^{p^a} = 1, \tau^{-1}\sigma\tau = \sigma^l, l = 1 + p^r + p^s \rangle$.

Define $X = \sum_{0 \le j \le p^{b+r}-1} \zeta_{p^{b+r}}^{-j} x(\sigma^j)$, $V_i = \tau^i X$ for $0 \le i \le p^a - 1$. It follows that

$$\begin{aligned} \sigma &: V_i \mapsto \zeta_{p^{b+r}}^{l^i} V_i, \\ \tau &: V_0 \mapsto V_1 \mapsto \cdots \mapsto V_{p^a-1} \mapsto V_0. \end{aligned}$$

Note that $K(V_0, V_1, \ldots, V_{p^a-1})^{\widetilde{G}}$ is rational by Theorem 2.7.

Define $U_i = V_i/V_{i-1}$ for $1 \le i \le p^a - 1$. Then $K(V_0, V_1, \ldots, V_{p^a-1})^{\widetilde{G}} = K(U_1, U_2, \ldots, U_{p^a-1})^{\widetilde{G}}(U)$ where

$$\begin{aligned} \sigma &: U_i \mapsto \zeta_{p^{b+r}}^{l^i - l^{i-1}} U_i,\ U \mapsto U \\ \tau &: U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^a-1} \mapsto (U_1 U_2 \cdots U_{p^a-1})^{-1},\ U \mapsto U. \end{aligned}$$

Notice that $\zeta_{p^{b+r}}^{l^i - l^{i-1}} = \zeta_{p^{b+r}}^{l^{i-1}up^r} = \zeta_{p^b}^{ul^{i-1}}$. Compare the actions of γ and α on $K(v_i : 1 \le i \le p^a - 1)$ with the actions of $\widetilde{G}$ on $K(U_i : 1 \le i \le p^a - 1)$. They are almost the same. (Indeed, there always exists z such that $uz \equiv x \pmod{p^b}$.) Hence, according to Theorem 2.7, we get that $K(v_1, \ldots, v_{p^a-1})^G(v) \cong K(U_1, \ldots, U_{p^a-1})^{\widetilde{G}}(U) = K(V_0, V_1, \ldots, V_{p^a-1})^{\widetilde{G}}$ is rational over K. Since by Lemma 2.4 we can linearize the action of α on $K(w_i : 1 \le i \le p^a - 1)$, we obtain finally that $K(w_i, v_i : 1 \le i \le p^a - 1)^{\langle\gamma,\alpha\rangle}$ is rational over K.

Sub-subcase IV.1.2 Let $s < r$. Then $l = 1 + v \cdot p^s$, where $v = 1 + p^{r-s}$. Consider the metacyclic p-group $\widetilde{G} = \langle \sigma, \tau : \sigma^{p^{b+s}} = \tau^{p^a} = 1, \tau^{-1}\sigma\tau = \sigma^l \rangle$. We can proceed in the same way as we did in the previous sub-subcase. (We need only to replace r with s and u with v.)

Subcase IV.2 Assume that $c-s>b$. We have that $(\zeta_{p^c}^{p^s k^{i-1}})^{xp^{c-b-s}} = \zeta_{p^{b+s}}^{xp^s k^{i-1}} = \zeta_{p^{b+s}}^{xp^s l^{i-1}} = \zeta_{p^b}^{xl^{i-1}}$, since $k^{i-1}p^s \equiv l^{i-1}p^s \pmod{p^{b+s}}$. For $1 \le i \le p^a-1$, define $w_i = v_i/u_i^{xp^{c-b-s}}$. Then $K(u_i, v_i : 1 \le i \le p^a-1) = K(u_i, w_i : 1 \le i \le p^a-1)$ and $\gamma(w_i) = w_i$ for $1 \le i \le p^a-1$.

Consider the metacyclic p-group $\widetilde{G} = \langle \sigma, \tau : \sigma^{p^c} = \tau^{p^a} = 1, \tau^{-1}\sigma\tau = \sigma^k, k = 1+p^s \rangle$. The proof henceforth is exactly the same as Case II.

4. Proof of Theorem 1.7

We keep the notations from the beginning of Section 3 and will proceed with the two cases.

Case I. Let $\varepsilon = 0$. The rationality problem for this group was investigated first by Chen [Ch] but with a mistake regarding the linearization technique. We will show that this error can be amended at the expense of the additional assumption $r = b-a$ given in the statement of our theorem.

Define $x_i = \alpha^i \cdot Y_1, y_i = \alpha^i \cdot Y_2$ for $0 \le i \le p^a-1$. Note that from the relations $[\beta, \alpha] = \gamma^{p^s}, [\gamma, \alpha] = \beta^{p^r}$ and the inequality $r+s \ge \max\{b, c\}$ it follows that $\beta\alpha^i = \alpha^i\beta\gamma^{ip^s}$ and $\gamma\alpha^i = \alpha^i\gamma\beta^{ip^r}$. We have now

$$\begin{aligned}
\beta \ &: x_i \mapsto \zeta_{p^c}^{ip^s} x_i,\ y_i \mapsto \zeta_{p^b} y_i,\\
\gamma \ &: x_i \mapsto \zeta_{p^c} x_i,\ y_i \mapsto \zeta_{p^b}^{ip^r} y_i,\\
\alpha \ &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^a-1} \mapsto x_0,\\
&\quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^a-1} \mapsto y_0.
\end{aligned}$$

for $0 \le i \le p^a-1$.

For $1 \le i \le p^a-1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \le i \le p^a-1) = K(x_0, y_0, u_i, v_i : 1 \le i \le p^a-1)$ and for every $g \in G$

$$g \cdot x_0 \in K(u_i, v_i : 1 \le i \le p^a-1) \cdot x_0,\ g \cdot y_0 \in K(u_i, v_i : 1 \le i \le p^a-1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \le i \le p^a-1)$ is invariant by the action of G. Thus $K(x_i, y_i : 0 \le i \le p^a-1)^G = K(u_i, v_i : 1 \le i \le p^a-1)^G(u, v)$ for some u, v such that $\alpha(v) = \beta(v) = \gamma(v) = v$ and $\alpha(u) = \beta(u) = \gamma(u) = u$. We have now

$$\beta \ : u_i \mapsto \zeta_{p^c}^{p^s} u_i,\ v_i \mapsto v_i,$$

$$\begin{aligned}
(4.1) \qquad \gamma\ &: u_i \mapsto u_i,\ v_i \mapsto \zeta_{p^b}^{p^r} v_i,\\
\alpha\ &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^a-1} \mapsto (u_1u_2\cdots u_{p^a-1})^{-1},\\
&\quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^a-1} \mapsto (v_1v_2\cdots v_{p^a-1})^{-1},
\end{aligned}$$

for $1 \le i \le p^a - 1$. From Theorem 2.2 it follows that if $K(u_i, v_i : 1 \le i \le p^a - 1)^G$ is rational over K, so is $K(x_i, y_i : 0 \le i \le p^a - 1)^G$ over K. Notice that $\zeta_{p^b}^{p^r} = \zeta_{p^a}$, since $a = b - r$.

Define $w_1 = v_1^{p^a}$ and $w_i = v_i/v_{i-1}$ for $2 \le i \le p^a - 1$. Then $K(u_i, w_i : 1 \le i \le p^a - 1) = K(u_i, v_i : 1 \le i \le p^a - 1)^{\langle\gamma\rangle}$ and the action of α on $w_i (1 \le i \le p^a - 1)$ is

$$\begin{aligned}
\alpha\ &: w_1 \mapsto w_1 w_2^{p^a},\\
&\quad w_2 \mapsto w_3 \mapsto \cdots \mapsto w_{p^a-1} \mapsto (w_1 w_2^{p^a-1} w_3^{p^a-2} \cdots w_{p^a-1}^2)^{-1} \mapsto\\
&\quad \mapsto w_1 w_2^{p^a-2} w_3^{p^a-3} \cdots w_{p^a-2}^2 w_{p^a-1} \mapsto w_2.
\end{aligned}$$

Define $z_1 = w_2, z_i = \alpha^i \cdot w_2$ for $2 \le i \le p^a - 1$. Now the action of α is

$$\alpha\ : z_1 \mapsto z_2 \mapsto \cdots \mapsto z_{p^a-1} \mapsto (z_1 z_2 \cdots z_{p^a-1})^{-1}.$$

Since $w_1 = (z_{p^a-1} z_1^{p^a-1} z_2^{p^a-2} \cdots z_{p^a-2}^2)^{-1}$, we get that $K(w_1, \ldots, w_{p^a-1}) = K(z_1, \ldots, z_{p^a-1})$. From Lemma 2.4 it follows that the action of α on $K(z_1, \ldots, z_{p^a-1})$ can be linearized.*

In this way we reduce the rationality problem of $K(u_i, v_i : 1 \le i \le p^a - 1)^G$ to the rationality of $K(u_i : 1 \le i \le p^a - 1)^{\langle\beta,\alpha\rangle}$ over K.

Compare Formula (4.1) (more specifically, the actions of β and α on $K(u_i : 1 \le i \le p^a - 1)$) with Formula (3.1) (the actions of γ and α on $K(v_i : 1 \le i \le p^a - 1)$). They are almost the same. Therefore, we can repeat the argument given after (3.1).

Case II. Let $\varepsilon = 1$. Define $x_i = \alpha^i \cdot Y_1, y_i = \alpha^i \cdot Y_2$ for $0 \le i \le p^a - 1$. Calculations show that $\gamma\alpha^i = \alpha^i \gamma^{l(i)} \beta^{ip^r}$ for $l(i) = (1 + p^t)^i$, and $\beta\alpha^i = \alpha^i \beta \gamma^{k(i)p^s}$ for $k(i) = \binom{i}{1} + \binom{i}{2}p^t + \cdots + \binom{i}{i}p^{(i-1)t}$. We have now

$$\begin{aligned}
\beta\ &: x_i \mapsto \zeta_{p^c}^{k(i)p^s} x_i,\ y_i \mapsto \zeta_{p^b} y_i,\\
\gamma\ &: x_i \mapsto \zeta_{p^c}^{l(i)} x_i,\ y_i \mapsto \zeta_{p^b}^{ip^r} y_i,\\
\alpha\ &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^a-1} \mapsto x_0,
\end{aligned}$$

*Note that, if we remove the assumption $a = b - r$, then $\zeta_{p^b}^{p^r}$ is not a primitive p^a-th root of unity and the order of γ is not p^a. If we try to apply the same linearization method, we will only get that $w_1^{p^{a-b+r}}$ is in $K(z_1, \ldots, z_{p^a-1})$, but w_1 may not be there. Therefore, we can not apply the same method in this situation.

$$y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^a-1} \mapsto y_0.$$

for $0 \le i \le p^a - 1$.

For $1 \le i \le p^a - 1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. Thus $K(x_i, y_i : 0 \le i \le p^a - 1) = K(x_0, y_0, u_i, v_i : 1 \le i \le p^a - 1)$ and for every $g \in G$

$$g \cdot x_0 \in K(u_i, v_i : 1 \le i \le p^a - 1) \cdot x_0,\ \ g \cdot y_0 \in K(u_i, v_i : 1 \le i \le p^a - 1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \le i \le p^a - 1)$ is invariant by the action of G. Thus $K(x_i, y_i : 0 \le i \le p^a - 1)^G = K(u_i, v_i : 1 \le i \le p^a - 1)^G(u, v)$ for some u, v such that $\alpha(v) = \beta(v) = \gamma(v) = v$ and $\alpha(u) = \beta(u) = \gamma(u) = u$. Notice that $k(i) - k(i-1) = l(i-1)$. We have now

$$\begin{aligned}
\beta\ &: u_i \mapsto \zeta_{p^c}^{l(i-1)p^s} u_i,\ \ v_i \mapsto v_i,\\
\gamma\ &: u_i \mapsto \zeta_{p^c}^{l(i-1)p^t} u_i,\ \ v_i \mapsto \zeta_{p^b}^{p^r} v_i,\\
\alpha\ &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^a-1} \mapsto (u_1 u_2 \cdots u_{p^a-1})^{-1},\\
&\quad\ v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^a-1} \mapsto (v_1 v_2 \cdots v_{p^a-1})^{-1},
\end{aligned}$$

for $1 \le i \le p^a - 1$. From Theorem 2.2 it follows that if $K(u_i, v_i : 1 \le i \le p^a - 1)^G$ is rational over K, so is $K(x_i, y_i : 0 \le i \le p^a - 1)^G$ over K.

Subcase II.1. $s \le t$. Instead of γ, we can take $\gamma_1 = \gamma\beta^{-p^{t-s}}$ with the action

$$\gamma_1\ : u_i \mapsto u_i,\ \ v_i \mapsto \zeta_{p^b}^{p^r} v_i,$$

for $1 \le i \le p^a - 1$. Similarly to Case I, we can reduce the rationality problem of $K(u_i, v_i : 1 \le i \le p^a - 1)^G$ to the rationality of $K(u_i : 1 \le i \le p^a - 1)^{\langle\beta,\alpha\rangle}$ over K. Compare the actions of β on $K(u_i : 1 \le i \le p^a - 1)$ with Formula (3.2) (more specifically, the actions of γ on $K(u_i : 1 \le i \le p^a - 1)$). They are almost the same. Apply the proof of Case II, Section 3.

Subcase II.2. $c \le \min\{2t, a + t\}$. Then $l(i-1)p^t \equiv p^t \pmod{p^c}$. For $1 \le i \le p^a - 1$ define $w_i = u_i/v_i^{p^{a-c+t}}$. We have now

$$\begin{aligned}
\beta\ &: w_i \mapsto \zeta_{p^c}^{l(i-1)p^s} w_i,\ \ v_i \mapsto v_i,\\
\gamma\ &: w_i \mapsto w_i,\ \ v_i \mapsto \zeta_{p^b}^{p^r} v_i,\\
\alpha\ &: w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{p^a-1} \mapsto (w_1 w_2 \cdots w_{p^a-1})^{-1},\\
&\quad\ v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^a-1} \mapsto (v_1 v_2 \cdots v_{p^a-1})^{-1},
\end{aligned}$$

for $1 \le i \le p^a - 1$. The proof henceforth is the same as Subcase II.1.

5. Proof of Corollary 1.8

Let V be a K-vector space whose dual space V^* is defined as $V^* = \bigoplus_{g\in G} K \cdot x(g)$ where G acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in G$. Thus $K(V)^G = K(x(g) : g \in G)^G = K(G)$.

Define $X_1, X_2, \ldots, X_s \in V^*$ by

$$X_j = \sum_{\ell_1,\ldots,\ell_s} x\left(\prod_{i\neq j} \alpha_i^{\ell_i}\right), \quad \text{for } 1 \le j \le s.$$

Note that $\alpha_i \cdot X_j = X_j$ for $j \neq i$. Let $\zeta_{p^{a_i}} \in K$ be a primitive p^{a_i}-th root of unity for $1 \le i \le s$. Define $Y_1, Y_2, \ldots, Y_s \in V^*$ by

$$Y_i = \sum_{m=0}^{p^{a_i}-1} \zeta_{p^{a_i}}^{-m} \alpha_i^m \cdot X_i$$

for $1 \le i \le s$.

It follows that

$$\alpha_i \ : Y_i \mapsto \zeta_{p^{a_i}} Y_i, \ Y_j \mapsto Y_j, \text{ for } j \neq i.$$

Thus $V_1 = \bigoplus_{1\le j\le s} K \cdot Y_j$ is a faithful representation space of the subgroup $H = \langle \alpha_1, \ldots, \alpha_s \rangle$.

Define $x_{ji} = \alpha^i \cdot Y_j$ for $1 \le j \le s, 0 \le i \le p^a - 1$. Recall that $[\alpha_i, \alpha] = \alpha_{i+1}^{p^{r_{i+1}}}$ for $1 \le i \le s - 1$. Hence

$$\alpha^{-i} \alpha_j \alpha^i = \alpha_j \alpha_{j+1}^{ip^{r_{i+1}}}, \quad \text{for } 1 \le j \le s-1, 1 \le i \le p^a - 1.$$

It follows that

$$\alpha_j \ : x_{ji} \mapsto \zeta_{p^{a_j}} x_{ji}, \ x_{j+1i} \mapsto \zeta_{p^{a_{j+1}}}^{ip^{r_{j+1}}} x_{j+1i}, \ x_{ki} \mapsto x_{ki}, \text{ for } k \neq j, j+1,$$

where $0 \le i \le p^a - 1$ and $1 \le j \le s - 1$.

Next, we need to consider four cases determined by the values of ε_1 and ε_2.

Case I. Let $\varepsilon_1 = 1, \varepsilon_2 = 0$. We have now

$$\begin{aligned} \alpha_s \ &: x_{1i} \mapsto \zeta_{p^{a_1}}^{ip^{r_1}} x_{1i}, \ x_{si} \mapsto \zeta_{p^{a_s}} x_{si}, \ x_{ki} \mapsto x_{ki}, \text{ for } k \neq 1, s, \\ \alpha \ &: x_{j0} \mapsto x_{j1} \mapsto \cdots \mapsto x_{jp^a-1} \mapsto x_{j0}, \text{ for } 1 \le j \le s, \end{aligned}$$

where $0 \le i \le p^a - 1$.

For $1 \leq j \leq s$ and for $1 \leq i \leq p^a - 1$ define $y_{ji} = x_{ji}/x_{ji-1}$. Thus $K(x_{ji} : 1 \leq j \leq s, 0 \leq i \leq p^a - 1) = K(x_{j0}, y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1)$, and for every $g \in G$

$$g \cdot x_{j0} \in K(y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1) \cdot x_{j0}, \text{ for } 1 \leq j \leq s$$

while the subfield $K(y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1)$ is invariant by the action of G, i.e.,

$$\begin{aligned}
\alpha_j &: y_{j+1i} \mapsto \zeta_{p^{a_{j+1}}}^{p^{r_{j+1}}} y_{j+1i}, \ y_{ki} \mapsto y_{ki}, \text{ for } k \neq j+1, 1 \leq j \leq s-1, \\
\alpha_s &: y_{1i} \mapsto \zeta_{p^{a_1}}^{p^{r_1}} y_{1i}, \ y_{ki} \mapsto y_{ki}, \text{ for } k \neq 1, \\
\alpha &: y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp^a-1} \mapsto (y_{j1} \cdots y_{jp^a-1})^{-1}, \text{ for } 1 \leq j \leq s,
\end{aligned}$$

where $0 \leq i \leq p^a - 1$.

From Theorem 2.2 it follows that if $K(y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1)^G$ is rational over K, so is $K(x_{j0}, y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1)^G$ over K.

Since $a_{j+1} - r_{j+1} = a$ for $1 \leq j \leq s-1$, we can apply $s-1$ times the technique explained in the proof of Case I, Theorem 1.7. In this way, we see that the action of α on $K(y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1)^{\langle \alpha_1, \ldots, \alpha_{s-1} \rangle}$ can be linearized. It remains to apply the final part of the argument of the mentioned proof in order to show that $K(y_{1i} : 1 \leq i \leq p^a - 1)^{\langle \alpha_s, \alpha \rangle}$ is rational over K.

Case II. Let $\varepsilon_1 = 0, \varepsilon_2 = 1$. Keeping the notations from Case I, we have

$$\alpha_s : y_{si} \mapsto \zeta_{p^{a_s}}^{p^{r_{s+1}}} y_{si}, \ y_{ki} \mapsto y_{ki}, \text{ for } k \neq s.$$

Since $r_{s+1} \geq r_s$, we get that α_s acts on $K(y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1)$ in the same way as $\alpha_{s-1}^{p^{r_{s+1}-r_s}}$. Then $K(y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1)^G = (K(y_{ji} : 1 \leq j \leq s, 1 \leq i \leq p^a - 1)^{\langle \alpha_1, \ldots, \alpha_{s-1} \rangle})^{\langle \alpha \rangle}$ is rational over K, according to Case I.

Case III. Let $\varepsilon_1 = \varepsilon_2 = 1$. We have now

$$\alpha_s : y_{1i} \mapsto \zeta_{p^{a_1}}^{p^{r_1}} y_{1i}, \ y_{si} \mapsto \zeta_{p^{a_s}}^{p^{r_{s+1}}} y_{si}, \ y_{ki} \mapsto y_{ki}, \text{ for } k \neq 1, s.$$

Since $r_{s+1} \geq r_s$, we can replace α_s with $\beta = \alpha_s \alpha_{s-1}^{-p^{r_{s+1}-r_s}}$, where

$$\beta : y_{1i} \mapsto \zeta_{p^{a_1}}^{p^{r_1}} y_{1i}, \ y_{ki} \mapsto y_{ki}, \text{ for } k \neq 1.$$

In this way, we have again reduced the problem to the considerations in Case I.

Case IV. Let $\varepsilon_1 = \varepsilon_2 = 0$. This case is trivial, since $\alpha_s(y_{ji}) = y_{ji}$ for $0 \leq i \leq p^a - 1, 1 \leq j \leq s$.

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